"Objective" Ambiguity and Ex Ante Trade

Federica Ceron

Université Paris1 Panthéon-Sorbonne

UFR 02 Sciences Economiques - Master 2 Recherche:
Empirical and Theoretical Economics (ETE)

“Objective” Ambiguity and Ex Ante Trade

Présenté et soutenu par:
Federica Ceron

Dirigé par:
Prof. Jean-Marc Tallon

June 6, 2014

L'université de Paris 1 Panthéon Sorbonne n'entend donner aucune approbation, ni désapprobation aux opinions émises dans ce mémoire; elles doivent être considérées comme propre à leur auteur.

1 Introduction

The purpose of this project is to explore the implications of assuming that individuals have maxmin preferences incorporating objective but imprecise probabilistic information on the features of Pareto optimal allocations.

The decision-theoretic model here considered has been proposed by Gajdos, Hayashi, Tallon and Vergnaud (2008)[5] with the purpose of explicitly linking the information possessed by the decision maker (or her state of knowledge) to her set of priors; this relation between cognitive and behavioural (attitudes toward risk or uncertainty) elements governing an agent's choices is left unmodeled in the standard maxmin preferences (MMEU) as axiomatized by Gilboa and Schmeidler (1989)[7], which perhaps for this reason, coupled with the radical nature of the min operator, attracted some well-known criticisms concerning its extremely "pessimistic" character (see for instance Wakker (2010), p. 325 [11]). While this criticism seems to be somewhat misplaced, since the set of priors is derived from preferences in the MMEU representation, and never identified with some set of probabilities compatible with the information available to the subject[1], the purely subjective nature of the set of priors reduces the model's descriptive power, at least in the sense of impeding statements like "the agent is pessimistic" or "more pessimistic than". The model discussed here provides the needed structure to allow such considerations, thereby enhancing the analytical possibilities in an economy-wide set-up. In particular, the reference to the set of probabilistic information available in the economy allows to determine an agent's *degree* of uncertainty aversion on the basis of which (or "how much") priors enters her final set of beliefs which will be relevant to her decisions[2]. Hence, the investigation of the relation between degrees of "pessimism" and trading behaviour seems to be meaningful.

With Bayesian agents Pareto optimality dictates that at equilibrium either there is risk sharing, when agents agree in their probabilistic assessment, or there be betting behavior, in case of disagreement, suggesting the existence of a link between common priors and (purely) speculative trade in the absence of uncertainty. Such a relationship holds in general when ambiguity is introduced and agents maximize (a broad class of) multiple priors models with convex preferences, provided that the aggregate endowment is constant across states. In particular, the Pareto locus coincides with the one of full insurance allocations whenever agents share at least one prior, and it involves only bets otherwise. This minimal form of agreement required suggests that uncertainty aversion may hamper trading opportunities in the sense of causing some insurance markets to

[1] Gilboa and Schmeidler proposed a second argument against such criticism, which is based on the axiomatic character of their derivation: "if you accept the axioms, you agree that people behave *as if* they had such a set C"(Gilboa, (2009), p. 164)[6].

[2] A different approach has been taken by Klibanoff, Marinacci and Mukerji (2005)[8], who replace the minimum expected utility with some second order probability. By introducing a second expected-utility-like functional form they are also able to capture the DM's attitude towards ambiguity while separating them by the risk ones. However, they do not model probabilities as objective entities.

break down, since DMs may be unwilling to insure each other. This intuition, however, has not been in general established, for when the aggregate endowment is not constant (a situation known as aggregate uncertainty) rank-dependent models typically give raise to two orders of difficulties: on one hand the prior relevant to the decision is in general allocation-specific, rendering its identification problematic; on the other, it needs not to be unique, making kinks arise robustly in the agents upper contour set.

The additional structure of the subclass of MMEU preferences here adopted motivates the present investigation, whose main aim is to verify whether, by exploiting the intrinsic comparability across agents it provides, those difficulties can be partially overcome, and the Pareto locus characterized. The existence of some set of objective evidence common to all agents, together with the specific mechanism through which each agent "selects" her set of priors, makes the intersection of the agents' set of priors always non-empty, thereby establishing the equivalence between efficiency and full insurance behavior under aggregate certainty. Therefore, the present work focuses solely on the case of aggregate uncertainty.

It is shown that indeed the Pareto locus can be characterized when there are just two states of the world: efficiency implies comonotonicity, a result which is not *per se* new in the literature, but also some sort of efficient ambiguity-sharing behavior; if there are more states, it cannot in general: the consideration of some specific subclasses of the aggregate endowment seems to be required. With respect to a previous result by Chateauneuf and De Castro (2011)[3], we study one case in which the simple observation ("by eye") of the aggregate endowment may be sufficient to know whether Pareto optima are guaranteed to be comonotone.

The rest of this work is organized as follows: the next paragraph offers a short overview of the literature; section 2 presents the set-up considered and defines the agents' preferences; in section 3 we briefly recall the properties of the efficient risk-sharing behaviour in the benchmark case of a von Neumann-Morgenstern economy, and present some notions which will be widely used in the analysis. Section 4 considers the risk sharing behavior of agents *à la* Gadjos et al. and derives the main results. Section 5 concludes.

1.1 Related literature

There is a broad literature on the general equilibrium implications of multiple priors models. The present analysis builds in particular upon the work on Pareto optimality and ex ante trade for Choquet expected utility (CEU) and MMEU preferences (recall that they coincide when the Choquet capacity is convex). In particular: Billot, Chateauneuf, Gilboa, and Tallon (2000)[1] show that in an economy of MMEU maximizers aggregate certainty implies that P.O. allocations involve full insurance and are therefore located on the common certainty line. Chateauneuf, Dana and Tallon (2000)[2] consider CEU preferences and show that when agents have the same convex capacity, the set of Pareto-optima is identical to the set of optima of an economy in which agents are expected utility maximizers and have the same probability. When agents have different capacities, the Pareto locus cannot in general be characterized, unless Pareto-optima are comonotone, requirement which is not guaranteed to hold. Comonotonicity of Pareto-optima helps characterizing the Pareto locus, although not fully. A full characterization is possible in the two-state case, where comonotonicity of the optima results if the intersection of the core of agents' capacities is non-empty; however, the risk sharing arrangement cannot be specified unless an economy with just two agents is considered, in which case the less pessimistic agent insures the other. Finally, they show that under aggregate certainty, the "common prior" condition (namely, a non-empty core intersection) is enough to guarantee that Pareto optimal allocations involve full insurance. Rigotti, Shannon and Strzalecky (2008)[9] show that the link between efficiency and common priors follows from the Welfare Theorems relating Pareto optimality to the existence of linear functionals providing a common support to the agents' preferred sets, coupled with the particular form these supports take for various classes of preferences. Chateauneuf and De Castro (2011)[3] find sufficient requirements for the relation "more ambiguity implies less ex ante trade" to be true. In particular, they define a notion of "unanimously unambiguous" allocation which parallels the one of full insurance in the case of aggregate risk. Even though the identification of P.O. and full insurance sets is lost, their no trade results extends the characterization of the Pareto locus to (some) situations of aggregate uncertainty, in a rather intuitive way. A similar approach based on priors conditional on unambiguous events is proposed by Strzalecki and Werner (2011) [10].

2 Set Up

2.1 The economy

We consider a standard two-period pure exchange economy with uncertainty in the second period, both for the agents' preferences and for the aggregate endowment. That is: (i) agents are unable to uniquely quantify the likelihood of the potential states of the world, and (ii) the aggregate endowment is non-constant across the possible states of the world.

It is assumed that there is only one good (say, money) and a finite number k of states of the world, indexed by the subscript $s = \{1, ..., k\}$. The set of states of the world will be denoted $S = \{1, ..., k\}$, while the aggregate endowment as $\omega = (\omega_1, ..., \omega_k)$.

There are n agents, indexed by $i = \{1, ..., n\}$. An action of agent i is a function $C^i : S \to \mathbb{R}$ which specifies the different levels of consumption associated to different states of the world; letting C^i_s be agent i's (state-contingent) consumption in state s, we denote by $C^i = (C^i_1, ..., C^i_k)$ a consumption asset of agent i.

An allocation $C = (C^i)_{i=1,...,n}$ is *interior* if $C^i_s > 0, \forall i, \forall s$ and it is *feasible* if $\sum_{i=1}^n C^i = \omega$.

Probabilistic information is a primitive of the economy, and is assumed to take the form of a probability-possibility set, that is, a set P of probability measures over the state space. The decision maker is told that the true probability law lies in P. Hence, information is "objective" in the sense that it is true, but may be imprecise or ambiguous. Since we interpret P as some objective evidence, it is natural to assume that all agents share the same information.

In a sense that will be further developed in the next section, each agent is assumed to "select" from the probability-possibility set P, which is exogenously given and describes the available information, her set of revealed priors, describing her probabilistic beliefs. The set of representation priors, called *selected* probability-possibility set, is constituted as a selection mapping from the probability-possibility set $\varphi_i : P \to P$ and denoted $\varphi_i(P) \subseteq P$.

The last definition is the one of full insurance allocation: an allocation C^i is said to involve *full insurance* for agent i if it assigns constant consumption (to i) across all states of the world, i.e. $C_1^i = ... = C_k^i$.

2.2 Consumers' Behaviour

Consumers are assumed to have preferences *à la* Gajdos et al. (2008) [5]; such preferences extend Gilboa's and Schmeidler's Multiple Priors by explicitly incorporating the information possessed by the DM as a primitive of the model.

The DM is assumed to have $\succeq$ over couples (P, C) where P is the probability-possibility set (a set of probability distribution over the state space) and C a consumption choice (act). Each agent's $i = 1, ..., n$ preferences are represented by a utility function $V^i : (\mathcal{P} \times \mathbb{R}^k)$ that takes the form:

$$
\begin{aligned}
V^i(P, C_i) &= \min_{\forall \pi \in \varphi_i(P)} \mathbb{E}_\pi U^i(C^i) \\
&= \min_{\forall \pi \in \varphi_i(P)} \sum_{s=1}^{k} \pi_s U^i(C_s^i)
\end{aligned}
\tag{2.2.1}
$$

Where:

- $U^i : \mathbb{R}^k \to \mathbb{R}$ is a utility index defined up to a positive linear transformation;
- $\varphi : \mathcal{P} \to \mathcal{P}$ is a (unique) mapping that makes explicit the link between the objective information possessed by the DM (P) and her "set of priors" ($\varphi(P)$).

Remark 1. *That preferences are defined over such pairs implies that, at least conceptually, consumers are allowed to compare the same act under different informational settings. In particular, for two PPSs P and Q and two acts $\overline{C}$ and $\hat{C}$, the DM is said to prefer (P, $\overline{C}$) to (Q, $\hat{C}$) if and only if:*

$$\min_{\forall p \in \varphi(P)} \mathbb{E}_p U(\overline{C}) \geq \min_{\forall q \in \varphi(Q)} \mathbb{E}_q U(\hat{C}) \tag{2.2.2}$$

Whenever the PPS is fixed, the above expression reduces to Maxmin Expected Utility (MEU), where the prior information is exogenously given and left unmodeled; the peculiar aspect of the model here considered is, however, that the link between the information possessed by the decision maker and the "set of priors" is made explicit through the mapping $\varphi(\cdot)$.

The above representation result is flexible as to which form $U^i(\cdot)$ can take. This is a merit of the model, since it allows to separate a DM's attitudes towards risk (embedded in the mapping $U^i(\cdot)$) from her attitudes towards informational precision (embedded in the mapping $\varphi_i(\cdot)$). For our purposes, however, we need to assume the following:

Assumptions U $U^i(\cdot)$ is $\mathcal{C}^1$, strictly increasing and strictly concave.

It remains to specify the functional form we wish to impose on $\varphi_i(\cdot)$. Recall that this mapping represents the DM's attitudes toward imprecision, and therefore constitutes the novelty of the proposed model. Investigating the link between P and $\varphi_i(\cdot)$, Gajdos *et al.* propose a characterization of $\varphi_i(\cdot)$ as a *contraction mapping*[3]. The selected PPS is obtained by (1) solving for the "*center*" of the probability-possibility set, that is, for some "salient" prior, and (2) shrinking the set towards the center to a degree given by the subjective preferences. Formally, the specific form taken by $\varphi_i(\cdot)$ is given by:

$$\varphi_i(P) = (1 - \varepsilon^i)s(P) + \varepsilon^i P \tag{2.2.3}$$

where:

i. s(P) is the Steiner point of the probability-possibility set P;

ii. $\varepsilon^i \in [0, 1]$ is a (subjective) index of aversion towards imprecision.

This characterizations assumes that the DM is averse to informational imprecision[4], that is, roughly, that she always prefers to act in a setting where she possesses more precise

[3]The authors provide also a second, more general, representation result (Theorem2), which is not very specific on the functional form of the mapping $\varphi_i(\cdot)$; examples compatible with theorem 2 are the identity mapping or entropy. However, the contraction representation is behaviourally appealing, and has the additional desiderata of providing a rather tractable interpersonal measure of the degree of aversion to imprecision. As an historical byproduct, it also formalizes Ellsberg's intuition [4] that "*To choose on a "maximin" criterion alone would be to ignore entirely those probability judgments for which there is evidence. But in situations of high ambiguity, such a criterion may appeal to a conservative person as deserving some weight, when interrogation of his own subjective estimates of likelihood has failed to disclose a set of estimates that compel exclusive attention in his decision-making*"(p. 662). Ellsberg also provides the example of a linear combination of "pure" maxmin criterion and some evidence-based probability estimate.

[4]More precisely, the contraction result is delivered in a representation theorem which assumes *Uncertainty Aversion* (as in MMEU) in the set of axioms defining the model-specific rationality; however, the authors also show that *Uncertainty Aversion* is implied by an axiom of *Aversion toward Imprecision* which compares the same act under different PPSs and roughly says that the DM always prefers to act in a setting where she possesses more precise information. Therefore, in the context of this analysis, aversion towards "imprecision" or "uncertainty" will denote the same behavioural attitude.

information. The parameter $\varepsilon^i \in [0,1]$ represents the DM's aversion towards imprecision in the sense that it determines her "proximity" to the behaviour of SEU maximization (or, equivalently, her degree of pessimism). It also allows to define a very tractable measure of comparative imprecision aversion:

Definition 1. *Comparative Imprecision Aversion:*
A DM "b" is said to be more averse to informational imprecision than a DM "a" if and only if $\varepsilon^b > \varepsilon^a$. A lower degree of aversion to imprecision results in a smaller set of beliefs: $\varphi^a(P) \subset \varphi^b(P)$.

Note that when the parameter ε^i equals 0, the contraction mapping will be a singleton, meaning that DM selects a unique prior from her PPS and can therefore be viewed as a SEU maximizer. This is true with the caveat that the selected prior cannot be *any* prior in the information set, but is restricted to be its Steiner point. Hence, when a DM is neutral with respect to informational imprecision, she will evaluate all acts according to the central tendency of the available information. In this sense even if all agents in the economy are uncertainty neutral, efficiency will exclude betting behavior. Further, when ε^i equals 1, the contraction mapping will be trivial in the sense of taking the form of an identity mapping: the DM will consider all priors in the PPS and therefore behave as a MMEU maximizer. Again, the set of beliefs is restricted to coincide with the PPS, implying that an economy of "pessimistic" agents will be homogeneous in terms of probabilistic assessment. Finally, note that when the PPS is itself a singleton, all agents will hold the same belief *regardless* of their subjective degree of uncertainty aversion.

Given the central role played by the parameter ε, (and in order to distinguish them from standard MMEU) in what follows we will denote preferences *à la* Gajdos *et al.* as ε-MMEU preferences.

The next example illustrates the functioning of the contraction mapping for different degrees of uncertainty aversion:

Example 1. Suppose that there are two states of the world and the information available is given by the set:

$$P = \{(\tfrac{1}{4}+\alpha, \tfrac{3}{4}-\alpha), \alpha \in [0, \tfrac{1}{2}]\}$$

For a given degree of aversion to imprecision ε^i, agent i's selected PPS is given by:

$$\phi^i(P) = \{(\tfrac{1}{4}+\beta, \tfrac{3}{4}-\beta), \beta \in [\tfrac{1}{4}-\theta, \tfrac{1}{4}+\theta], |\beta| \leq \alpha\}$$

where $\beta = 1 - \varepsilon^i, \forall i$.

Consider three agents, A, B and C, with increasing degree of aversion to imprecision: $\varepsilon^A = 0 < \varepsilon^B = \frac{2}{3} < \varepsilon^C = 1$. The agents' selcted PPS are:

$$\phi^A(P) = s(p) = \{(\tfrac{1}{2}, \tfrac{1}{2})\}$$
$$\phi^B(P) = co\{(\tfrac{1}{3}, \tfrac{2}{3}), (\tfrac{2}{3}, \tfrac{1}{3})\}$$
$$\phi^C(P) = P = co\{(\tfrac{1}{4}, \tfrac{3}{4}), (\tfrac{3}{4}, \tfrac{1}{4})\}$$

Note that $\phi^A(P) \subset \phi^B(P) \subset \phi^C(P)$.

Figure 1 represents the indifference curves of the three agents in example 1: it is easy to see that uncertainty aversion increases the "concavity" of the most preferred set, since the two segments of the indifference curves (the one that lies above and below the certainty line, respectively) are tangent, at any constant bundle, to the line whose slope is given by the ratio of probabilities of the two states, computed according to the minimizing priors. As it will be explain in detail, higher degree of uncertainty aversion increases the set of excluded trades from any constant bundle.

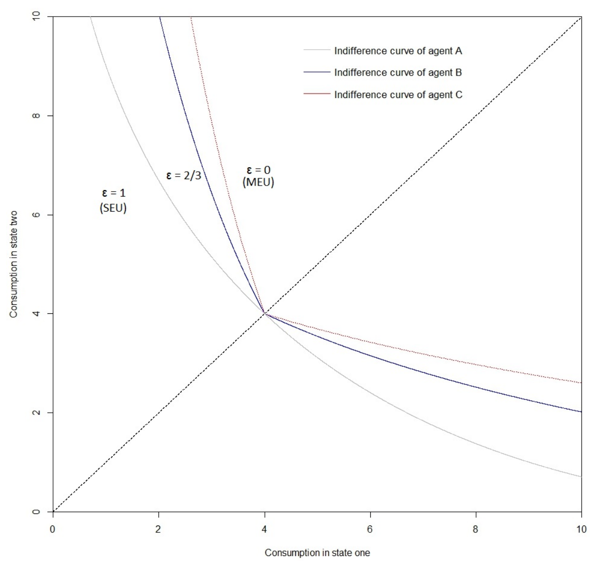

Figure 1: Different degrees of uncertainty aversion

2.3 Indifference curves and non-differentiability

In the SEU framework probability is identified with the odds at which a DM is willing to make small bets and can therefore be identified with an hyperplane supporting the upper contour set, as proposed by Yaari (1969)[12]. In multiple priors models, the upper contour set typically displays kinks, implying that there exist multiple supporting hyperplanes at some acts. This is precisely what happens with preferences of the ε-MMEU type: each agent considers the minimum expected utility over her selected probability possibility set, and this "minimum" generates kinks on the indifference curves when more than one probability realize the minimum. Hence, indifference curves are non-differentiable at some acts. In the the two state case kinks arise only on the certainty line, while when there are more than two states they may arise at non-constant acts too. This difference will be important in the analysis of the agents' market behaviour.

When there are just two states of the world, it is possible to identify the priors that minimize each agent's utility at all feasible allocations. This is because there are only two classes of allocations relevant to the decision process: (i) those yielding more consumption in state 1 than in state 2, and (ii) those yielding less[5]. Any allocations in (i) will be minimized by the same prior, namely the one assigning, among all priors in the agent's selected PPS, the *lowest* probability to the "good" outcome, i.e. state 1. Similarly, all allocation in (ii) will be minimized by the prior assigning the *highest* probability to state 1. Allocations that involve constant consumption across the states of the world and are therefore located on the 45-degree line are assigned the same level of utility according to all priors: when some level of consumption is given regardless of the states of the world, information about their likelihood is irrelevant. As a consequence, kinks of the indifference curves are only located on the certainty line.

On the other hand, when there are more than two states of the world, it is impossible to consider "types" of allocations: the minimizing prior at any bundle depends on the bundle itself, not solely on the ranking of outcomes. Further, there may be several priors which minimize the agent's utility at some (not necessarily constant) act. Whenever more than one probability realize the minimum, the upper contour set displays a kink.

We conclude this section by mentioning that the set $\varphi^i(P)$ captures two important aspects of market behavior. On one hand, agents are unwilling to trade from a constant bundle to a random one if the two have the same expected value for *some* prior in the set $\varphi^i(P)$. In particular, the set $\varphi^i(P)$ is the largest set of beliefs revealed by this unwillingness to trade based on zero expected net returns. On the other, agents are willing to trade from a constant bundle to a random one whenever the random act has greater expected value according to *every* prior in the set $\varphi^i(P)$. In particular, the set $\varphi^i(P)$ is the smallest set of beliefs revealing this willingness to trade based on positive expected net returns[6].

Consider again Figure 1 above. Now it becomes perhaps clearer why, in the two-state case, "higher degree of uncertainty aversion increases the set of excluded trades from any constant bundle": at any constant bundle, the slope of the indifference curve which lies above the 45-degree line is given by $\frac{p_1}{1-p_1}$,while the slope of the indifference curve lying below by $\frac{q_1}{1-q_1}$, where "p" and "q" denote the minimizing priors for acts yielding higher and lower consumption in state 1 than in state 2, respectively. A higher degree of pessimism correspond to a steeper (flatter) slope of the indifference curve above (resp. below) the certainty line. Hence, it rules out some trades which are theoretically possible for a more optimistic agent.

3 Pareto Optimality, Full Insurance and Betting

A betting situation occurs when, at the optimum, (possibly risk averse) agents find it mutually beneficial to engage in uncertainty-generating trades because they disagree on

[5]Putting it differently, only the *ranking* of outcomes matters.

[6]Rigotti *et al.*[9] shows that this property hold for several classes of convex preferences.

the likelihood of the different states of the world. Suppose for simplicity that there is no aggregate risk. There are two agents, Anne and Bob, endowed with a bond that pays €1 if the newly elected government manages to introduce the claimed reform of the pension system and if it doesn't. They are both risk averse but they strongly disagree on the likelihood of the reform being introduced, and therefore decide to make out of the bond a risky asset paying zero in the state which they judge as "less likely", respectively.

On the other hand, if the two agents agree on the likelihood of the two states, they will rather switch from a risky asset to a bond, thereby insuring each other.

These two typologies of behavior are linked to the notion of comonotonicity of the consumption plans:

Definition 2. *A family of random variables $(C^i)_{i=1,\dots,n}$ on S is a class of comonotone function if for all i, i' and for all s, s' , $(C^i_s - C^i_{s'})(C^{i'}_s - C^{i'}_{s'}) \geq 0$.*

The notion intuitively means that the random variables "vary in the same direction", and it has a natural interpretation in terms of mutualization of risk: whenever agents agree on the relative likelihood of the states of the world, they will choose a "similar" consumption pattern, sharing the aggregate risk efficiently on the basis of their risk attitudes. Otherwise, they will "bet" on the state which they judge as more likely, compared to the other agents' beliefs, i.e. they will choose non-comonotone consumption plans.

3.1 Optimal risk-sharing with vNM agents

When agents are vNM maximizers with utility indices $U^i, i = 1, ..., n$, the efficient locus can be fully characterized. In particular:

1. if agents have the same probability over the set of state of the world $\pi = (\pi_1, ..., \pi_k)$, $\pi_s > 0, \forall s$ and a utility function defined by $v^i(C^i) = \sum_{s=1}^k \pi_s U^i(C^i_s), i = 1, ..., n$, P.O. allocations are independent of the (common) probability, depend only on aggregate risk (and utility indices), and are comonotone. Consequently, agent i's consumption C^i is a non-decreasing function of the aggregate endowment ω.

2. if agents have different probabilities $\pi^i, i = 1, ..., n$, P.O. allocations do depend on the probabilities (and on aggregate risk and utility indices), and are not necessarily comonotone[7].

Example 2. As an illustration, consider the Edgeworth box for an economy with two vNM agents, two states and one good. The left panel of Figure 2 plots the contract curve in the case where agents have the same probability assessment, the right one when their priors differ. Note that in the box comonotone allocations belong to the region between the two agents' certainty lines.

[7]Whether there will be non-comonotone P.O. allocations depends on the utility indices and on the level of disagreement between the two agents' probabilistic assessment. The easiest example of non-comonotone P.O. locus is an economy with aggregate certainty and in which agents hold different beliefs. In this case each agent "bets" on the state which she judges more likely, relative to the other agent's beliefs.

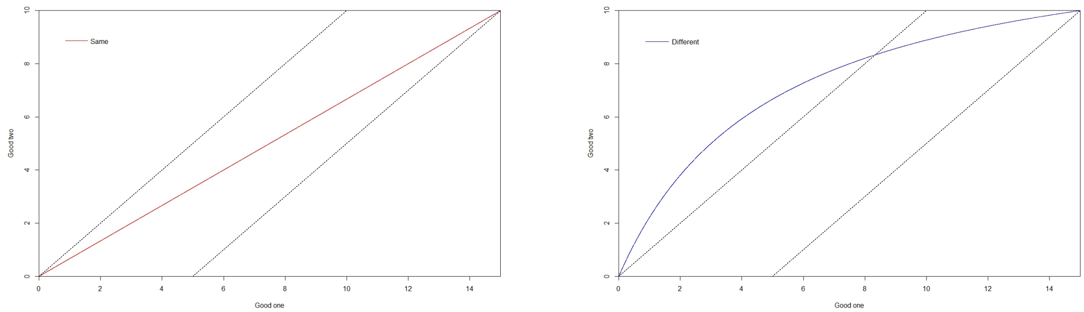

Figure 2: vNM economy with same (left) and different (right) probabilities

4 Optimal risk-sharing with ε-MMEU agents

Given the link between consensus in the probabilistic assessment and efficient risk sharing behavior, it seems plausible that ε-MMEU agents will bet less against each other than vNM maximizers: on one hand the objective information they share imposes some restrictions on their possible disagreement; on the other, they are averse to ambiguity. Indeed, this intuition turns out to be true in the two-state case, where Pareto optima are always comonotone and the risk-sharing arrangement is related to the relative degree of uncertainty aversion across agents; roughly, the less uncertainty averse agent insures the more. However, comonotonicity of the Pareto optima cannot be typically guaranteed when there are more than two states of the world, and the relation between attitudes towards uncertainty and risk taking behavior is lost. Nevertheless, for some classes of initial endowments the Pareto locus can be characterized, as shown by Chateauneuf and De Castro (2011). Compared to standard MMEU preferences, the objective character of the information available imposes to ε-MMEU agents some additional homogeneity in beliefs.

Before considering the two situations (two and more states of the world), we briefly recall how the the efficient locus is obtained.
The contract curve results by solving the utility maximization problem of the n agents, subject to the endowment constraints:

$$\text{(UMP):} \quad \begin{cases} \underset{\{(C^i)_{s=1,\ldots,k}\}}{\text{maximize}} & V^i(P, C^i) = \min\limits_{\forall \pi \in \varphi_i(P)} \mathbb{E}_\pi U^i(C^i) \quad, \forall i \\ \text{subject to} & \omega = \sum\limits_{i=1}^{n} C^i \end{cases}$$

The corresponding first order condition for an interior optimum, which, whenever U^i is concave, is necessary and sufficient, is obtained by equating the marginal rates of substitution between any two states across individuals. These MRS are weighted by the ratio of the (relevant) probabilities, as in the standard expected utility setting. Denoting with

$\pi^i \in \varphi^i(P)$ the prior that minimizes agent i's expected utility at the optimal allocation, the optimality condition is, $\forall (i,j) \in \mathcal{N}^2$, $\forall (s,s') \in S^2$:

$$\left(\frac{\pi^i_s}{\pi^i_{s'}}\right)\left(\frac{U'_i(C^i_s)}{U'_i(C^i_{s'})}\right) = \left(\frac{\pi^j_s}{\pi^j_{s'}}\right)\left(\frac{U'_j(C^j_s)}{U'_j(C^j_{s'})}\right) \tag{FOC}$$

Or, in a more compact way:

$$\cap_{i\in\mathcal{N}} P_i(C^i) \neq \emptyset, \quad \text{where:} \quad P_i(C^i) \equiv \left\{ U'_i(C_i)\pi : \pi \in arg \min_{\forall p \in \varphi_i(P)} \mathbb{E}_p U^i(C^i) \right\} \tag{FOC'}$$

The second expression makes it clear that, whenever multiple priors minimize the value of an act, making the upper contour set non-differentiable, we need to consider the *set* of marginal rates of substitution resulting by computing the MRS w.r.t. all the minimizing priors[8] at the particular allocation. The sets $P_i(C^i)$ in (FOC') have been interpreted (see for instance [9] and [3]), after appropriate normalization, as some "modified" set of subjective beliefs, so that the condition on non-empty intersection may be viewed as some sort of "agreement" among agents. However, as Chateauneuf and De Castro note, the existence of a common "modified" prior "is not completely satisfactory from a characterization point of view, because the probabilities in the sets $\mathcal{P}_i(C_i)$ are distorted by the marginal utilities of consumers. Therefore, these probabilities do not depend exclusively on the agents' beliefs"(p. 14 in [3]) . This is perhaps even more true in the setting here considered, since the agents' priors are linked to some core of objective evidence: once we allow the marginal utilities to enter the determination of the "modified" set of priors, this link is lost.

Finally, note that, by the assumption on (strict) concavity of the utility indices, the following property must hold:

$$\forall i, \quad C^i_1 > C^i_2 \Rightarrow U'_i(C^i_1) < U'_i(C^i_2) \tag{DMU}$$

4.1 The two-state case

In the two-state case, the fact that the minimizing priors will just depend on the *ranking* of outcomes strongly simplifies the analysis, since there are just two relevant priors, which can be compared in terms of the weight they assign to each state. In particular, denoting p^i and q^i the minimizing prior when C^i_1 and C^i_2 is the best outcome, respectively, we are guaranteed that p^i_1 won't be greater than q^i_1 (and $p^i_2 = 1 - p^i_1$ won't be lower than $q^i_2 = 1 - q^i_1$). Further, the contraction representation allows an interpersonal comparison of the minimizing priors: roughly, the more pessimistic an agent is, the lower the probability she puts on the "good" outcome. We formalize these properties:

$$\forall i, \quad p^i_1 \leq q^i_1 \Rightarrow \frac{p^i_1}{1-p^i_1} \leq \frac{q^i_1}{1-q^i_1} \Leftrightarrow \frac{q^i_1}{1-q^i_1}\frac{1-p^i_1}{p^i_1} \geq 1 \tag{P1}$$

[8] Equivalently, the convex hull thereof.

$$\varepsilon^i > \varepsilon^j \Rightarrow \begin{cases} \frac{p_1^i}{1-p_1^i} > \frac{p_1^j}{1-p_1^j} \\ \frac{q_1^i}{1-q_1^i} < \frac{q_1^j}{1-q_1^j} \end{cases} \tag{P2}$$

Finally, note that, given the simple form taken by the agents' utility function:

$$V^i(P, C^i) = \begin{cases} p_1^i U(C_1^i) + (1-p_1^i)U(C_2^i), & C_1^i \geq C_2^i \\ q_1^i U(C_1^i) + (1-q_1^i)U(C_2^i), & C_2^i > C_1^i \end{cases}$$

also the FOC simplifies:

$$\forall (i,j) \in \mathcal{N}^2, \qquad \left(\frac{\pi_1^i}{1-\pi_1^i}\right)\left(\frac{U_i'(C_1^i)}{U_i'(C_2^i)}\right) = \left(\frac{\pi_1^j}{1-\pi_1^j}\right)\left(\frac{U_j'(C_1^j)}{U_j'(C_2^j)}\right) \tag{FOC}$$

Note that at any allocation we know ex ante what prior will each agent select and what sign the marginal rate of substitution, net of the probability weight, will have. This analytical desiderata is at the core of the results provided in the next two paragraphs.

4.2 Same degree of imprecision aversion

When all agents have the same degree of aversion to imprecision, they will select from the PPS the same (two) minimizing priors for the regions where $C_1^i > C_2^i$ and where $C_1^i < C_2^i$. As a consequence, the economy reduces to a vNM economy where agents hold the same probability (or to any parallel MMEU economy where agents have the same set of priors). Hence, P.O. allocation will depend only on the utility indices and on the aggregate uncertainty.

Proposition 1. *Whenever $\varepsilon^i = \varepsilon^j, \forall (i,j) \in \mathcal{N} \times \mathcal{N}$, the P.O. locus coincides with the one of an economy where agents are vNM maximizers with utility indices U^i and identical probability over the set of state of the world*[9]

Proof. First note that when agents have the same index of imprecision aversion, they also have the same (two) minimizing priors, say p (when $C_1^i > C_2^i$) and q (when $C_1^i < C_2^i$).
Assume, by contradiction, that there exist an interior P.O. allocation in which some agents i consumes strictly more in state 1 while some other agent j in state 2. Since the allocation is P.O., the FOC must be satisfied:

$$\left(\frac{p_1}{1-p_1}\right)\left(\frac{U_i'(C_1^i)}{U_i'(C_2^i)}\right) = \left(\frac{q_1}{1-q_1}\right)\left(\frac{U_j'(C_1^j)}{U_j'(C_2^j)}\right)$$

However, by (DMU) and (P1) this is clearly impossible:

[9]Hence, P.O. allocations are comonotone, none of the agents fully insures, the locus is independent of the particular prior(s) and it depends only on the aggregate uncertainty and on the utility indices.

$$\Leftrightarrow \underbrace{\left(\frac{U_i'(C_1^i)}{U_i'(C_2^i)}\right)}_{<1} = \underbrace{\left(\frac{q_1}{1-q_1}\right)\left(\frac{1-p_1}{p_1}\right)}_{\geq 1}\underbrace{\left(\frac{U_j'(C_1^j)}{U_j'(C_2^j)}\right)}_{>1}$$

Therefore, non-comonotone allocations cannot be optimal.

Consider now comonotone allocations and assume w.l.o.g. that $\omega_1 > \omega_2$: by the endowment constraints optimal allocations will be s.t. all agents consume at least as much in state 1 than in state 2 (i.e. $\forall i, C_1^i \geq C_2^i$) and will thus refer to the same minimizing prior p. Hence, the probability ratios cancel out and the optimality condition reduces to the one of an economy of vNM maximizers with identical probabilistic assessment:

$$\cancel{\left(\frac{p_1}{1-p_1}\right)}\underbrace{\left(\frac{U_i'(C_1^i)}{U_i'(C_2^i)}\right)}_{\leq 1} = \cancel{\left(\frac{p_1}{1-p_1}\right)}\underbrace{\left(\frac{U_j'(C_1^j)}{U_j'(C_2^j)}\right)}_{\leq 1}$$

Consequently, the two loci (which, under assumptions U and C, are well-defined) coincide: allocations do not depend of the prior p but only on the aggregate uncertainty[10] and on the utility indices.

Note also that no agent fully insures at an interior optimum: DMs behave *as if* the whole uncertainty in the economy were reduced to some objective, and commonly known, risk (here, summarized by the vector of probabilities p). Since they all are risk averse (by assumption C), they will mutualize the (perceived) risk. To see this, assume by contradiction that agent i insures while some other agent $j \neq i$ does not (by the endowment constraint it is clearly impossible that all agents simultaneously insure). The FOC $(i, j \neq i)$ is clearly never satisfied:

$$1 \neq \underbrace{\left(\frac{U_j'(C_1^j)}{U_j'(C_2^j)}\right)}_{<1}$$

It is important to remark that the first order approach is not completely justified in the analysis of constant allocations, since the upper contour set is non differentiable. However, note that no other prior in agent i's selected PPS could rationalize the choice of a full insurance allocation since, by (P1), her MRS at the certainty line is the lowest when the bundle is evaluated w.r.t. the prior p:

$$\left(\frac{q_1}{1-q_1}\right)(1) \geq \left(\frac{p_1}{1-p_1}\right)(1) > \left(\frac{U_j'(C_1^j)}{U_j'(C_2^j)}\right)\left(\frac{p_1}{1-p_1}\right)$$

□

[10] That is, whenever $\omega_s > \omega_{s'}$, all agents will consume (strictly) more in state s than in state s'.

4.3 Different degrees of imprecision aversion

When agents have different degrees of uncertainty aversion, they "contract" the set of available information at different rates. The selected PPS of the more uncertainty averse agent will include the one of the less: $\varepsilon^i < \varepsilon^{i+1} \Leftrightarrow \varphi^i(P) \supset \varphi^{i+1}(P)$. Hence, their two minimizing priors will be different. Denote by p^i and q^i the minimizing prior of agent i when $\mathcal{C}_1^i > C_2^i$ and when $\mathcal{C}_1^i < C_2^i$.

Assume w.l.o.g. that agents are ordered in their degree of uncertainty aversion: $\varepsilon_1 < \varepsilon_2 < ... < \varepsilon_n$. As we mentioned at the beginning of this section, the comparability of the agents' selected PPS allows, in the 2-state case[11], to identify a property relating the minimizing priors of all agents. This property roughly states that if an agent is more averse to uncertainty than another, she is more "pessimistic" in the sense of putting lower probability on the "good" outcome:

$$\forall i, \quad \frac{p_1^i}{1-p_1^i} > \frac{p_1^{i+1}}{1-p_1^{i+1}} \tag{P2.a}$$

$$\forall i, \quad \frac{q_1^i}{1-q_1^i} < \frac{q_1^{i+1}}{1-q_1^{i+1}} \tag{P2.b}$$

Further, P1 still holds (with strict inequality for all agents $i \geq 2$ and weak for the least uncertainty averse agent, who possibly behaves as a SEU maximizer).

Combining (P.2a)-(P2.b), and (P1) we obtain the following condition:

$$\frac{q_1^n}{1-q_1^n} > ... > \frac{q_1^1}{1-q_1^1} \geq \frac{p_1^1}{1-p_1^1} > ... > \frac{p_1^n}{1-p_1^n} \tag{P2}$$

We can now characterize the P.O. locus:

Proposition 2. *Assume $\varepsilon^1 < \varepsilon^2 < ... < \varepsilon^n$, $k = 2$ and $\omega_1 > \omega_2$. Let assumption C and assumption U hold. Efficiency implies that:*

1. *Allocations are comonotone;*
2. *If it is optimal for some agent to fully insure, it must be the **most** imprecision averse agent in the economy, i.e. agent "n". It is never optimal for agents $i = 1, ..., n-1$ to fully insure*[12]*.*
3. *An "order" on the agents' "deterministic" marginal rates of substitutions (i.e. on the ratio of their marginal utilities) between consumption in the two states is imposed by the degree of aversion to imprecision. If tastes are homogeneous, this is equivalent to a risk taking behaviour which decreases with the degree of aversion to imprecision ε^i.*

[11] This cannot be generalized to n-states: even though the agents' selected PPS are ordered by set inclusion, priors cannot in general be compared, as it will be explained in section 4.

[12] Note that it is also impossible that more than one agent simultaneously insure.

Proof. We prove (1)-(3) in turns.

(1) Comonotonicity of P.O. allocation: recall that at an interior optimum the following condition must be satisfied $\forall i = 1, ..., n$:

$$\frac{\pi_1^{i-1}}{1-\pi_1^{i-1}}\left(\frac{U'_{i-1}(C_1^{i-1})}{U'_{i-1}(C_2^{i-1})}\right) = \frac{\pi_1^{i}}{1-\pi_1^{i}}\left(\frac{U'_{i}(C_1^{i})}{U'_{i}(C_2^{i})}\right) = \frac{\pi_1^{i+1}}{1-\pi_1^{i+1}}\left(\frac{U'_{i+1}(C_1^{i+1})}{U'_{i+1}(C_2^{i+1})}\right) \tag{FOC}$$

Now suppose that agent i consumes strictly more in state 1 than in state 2. By concavity of the utility indices and by (P2), it is clearly impossible that agent $i+1$ (resp. agent $i-1$) consumes strictly more in state 2 than in state 1, since:

$$\underbrace{\left(\frac{U'_{i}(C_1^{i})}{U'_{i}(C_2^{i})}\right)}_{<1} = \underbrace{\frac{1-p_1^i}{p_1^i}\frac{q_1^{i+1}}{1-q_1^{i+1}}}_{>1}\underbrace{\left(\frac{U'_{i+1}(C_1^{i+1})}{U'_{i+1}(C_2^{i+1})}\right)}_{>1} \tag{FOC(i, i+1)}$$

and

$$\underbrace{\left(\frac{U'_{i}(C_1^{i})}{U'_{i}(C_2^{i})}\right)}_{<1} = \underbrace{\frac{1-p_1^i}{p_1^i}\frac{q_1^{i-1}}{1-q_1^{i-1}}}_{>1}\underbrace{\left(\frac{U'_{i-1}(C_1^{i-1})}{U'_{i-1}(C_2^{i-1})}\right)}_{>1} \tag{FOC(i, i-1)}$$

Hence, P.O. allocation are comonotone.
(2) Noting first that P2 excludes the possibility that, at an optimum, more than one agent insures, suppose that some agent $i \neq 1$ fully insures.

$$\underbrace{\frac{1-p_1^i}{p_1^i}\frac{p_1^{i-1}}{1-p_1^{i-1}}}_{>1}\underbrace{\left(\frac{U'_{i-1}(C_1^{i-1})}{U'_{i-1}(C_2^{i-1})}\right)}_{<1} = 1 = \underbrace{\frac{1-p_1^i}{p_1^i}\frac{p_1^{i+1}}{1-p_1^{i+1}}}_{<1}\underbrace{\left(\frac{U'_{i+1}(C_1^{i+1})}{U'_{i+1}(C_2^{i+1})}\right)}_{<1} \tag{FOC}$$

While the first equality may be satisfied, the second one is never: more uncertainty averse agent do not insure less ones. Hence, if at some interior P.O. allocation some agent insure, it must be the most uncertainty averse agent in the economy (here, agent n).
(3) From the previous two results we know that at an interior P.O. allocation, agents will share the risk at a rate determined by their degree of uncertainty aversion and by the relative concavity of their utility indices. Since the attitude towards uncertainty allows one to compare the agents' minimizing priors, a resulting order is imposed on the DMs' MRS between the two states. In particular, keeping the assumption that $\omega_1 > \omega_2$, the optimality condition is:

$$\left(\frac{p_1}{1-p_1}\right)\left(\frac{U'_1(C_1^1)}{U'_1(C_2^1)}\right) = ... = \left(\frac{p_n}{1-p_n}\right)\left(\frac{U'_n(C_1^n)}{U'_n(C_2^n)}\right)$$

Therefore, by P2:

$$\left(\frac{p_1}{1-p_1}\right) > ... > \left(\frac{p_n}{1-p_n}\right) \Rightarrow \left(\frac{U_1'(C_1^1)}{U_1'(C_2^1)}\right) < ... < \left(\frac{U_n'(C_1^n)}{U_n'(C_2^n)}\right) (< 1) \tag{4.3.1}$$

The last condition does not characterize much the Pareto locus in an economy where there is heterogeneity in tastes. However, if all agents have the same utility index U, the above condition has a clear-cut interpretation in terms of risk sharing behaviour: while all agents consume more in the state where aggregate resources are more abundant, the less uncertainty averse agent consumes much more:

$$\left(\frac{U'(C_1^1)}{U'(C_2^1)}\right) < ... < \left(\frac{U'(C_1^n)}{U'(C_2^n)}\right) \Leftrightarrow (C_1 - C_2)_1 > (C_1 - C_2)_2 > ... > (C_1 - C_2)_n \geq 0 \tag{4.3.2}$$

□

The Pareto locus in this case departs from the one of a vNM economy where agents hold different probabilities.

Example 3. *Betting (vNM/MEU): Non-comonotone P.O. allocation* Consider a SEU economy with three agents, A, B and C. Assume that $U^i(\cdot) = \log(\cdot), \forall i$ and that the probabilities of each agent are given by: $\pi^A = (\frac{1}{2}, \frac{1}{2})$, $\pi^B = (\frac{1}{3}, \frac{2}{3})$ and $\pi^C = (\frac{1}{4}, \frac{3}{4})$.
The allocation $C^A = (6,3), C^B = (4,4), C^C = (2,3)$ is Pareto optimal and it does not involve comonotone consumption bundles across agents.

Note that this example is consistent with A,B and C being MEU maximizers whose minimizing prior is precisely π^i at the relevant allocation. For instance, assign to each agent the following sets of priors:

$$P^A = \{(\tfrac{1}{2} + \alpha, \tfrac{1}{2} - \alpha), \alpha \in [0, \tfrac{1}{2}]\}$$
$$P^B = \{(\tfrac{1}{3} + \beta, \tfrac{2}{3} - \beta), \beta \in [0, \tfrac{1}{2}]\}$$
$$P^C = \{(\tfrac{1}{4} - \gamma, \tfrac{3}{4} + \gamma), \gamma \in [0, \tfrac{1}{4}]\}$$

The absence of restrictions on the "disagreement" between agents may lead to the Pareto optimality of betting behavior in a MMEU economy, i.e. in an economy with uncertainty aversion. To get comonotonicity of the MMEU locus (in the two-state case) some "minimal consensus" is required, namely a non-empty intersection of the agents' set of priors. This is always guaranteed in the ε-MEU setting.

4.3.1 The 2-agent 2-state case

When there are just two agents, the simple form taken by the endowment constraints allows us to derive a further property of the Pareto locus:

Proposition 3. *Assume n=2, $k = 2$, $\omega_1 > \omega_2$ and $\varepsilon^A > \varepsilon^B$. Let assumption C and assumption U hold. Then Proposition 2 holds and one can distinguish two possibilities:*

(i) *whenever in the corresponding vNM economy the locus of P.O. allocations is fully included in region α, the two loci coincides;*

(ii) *if P.O. in the corresponding vNM economy implies the existence of non-comonotonic P.O. allocations, the subset(s) of the vNM P.O. locus where consumption assets are non-comonotonic are replaced by the corresponding segments along the diagonal of the more uncertainty averse agent, who therefore is fully insured by the other agent.*

The proof consists in showing that any non-comonotone P.O. allocation (that could be P.O. in a vNM economy) is Pareto dominated by a corresponding allocation which involves full insurance for the more uncertainty averse agent. To simplify the analysis, it is convenient to consider the Edgeworth Box and to divide it in three regions:

$$\alpha = \begin{cases} C_1^A > C_2^A \\ C_1^B > C_2^B \end{cases} \quad , \quad \beta = \begin{cases} C_1^A > C_2^A \\ C_1^B < C_2^B \end{cases} \quad , \quad \text{and} \quad \gamma = \begin{cases} C_1^A < C_2^A \\ C_1^B > C_2^B \end{cases}$$

Clearly, allocations are comonotone in region α and are not in regions β and γ.

Proof. It is easy to see that whenever allocations are comonotone, the optimality condition corresponds to the standard condition in the vNM economy where agent A holds beliefs p^A and agent B p^B. It is also impossible that a P.O. allocation lies on agent B's certainty line, since she is the least uncertainty averse agent. Finally, note that even in the corresponding vNM economy P.O. allocations cannot be found in region β, since agent B assigns higher probability to state 1, relative to the belief of agent A: if some "betting" is to occur, it must be in region γ.

It remains to show that any allocation in region γ that would be P.O. in a vNM economy cannot be such here, since it is Pareto dominated by a corresponding allocation on agent A's certainty line. Fix an arbitrary allocation (C^A, C^B) in region γ and recall that it is such that:

$$\gamma = \begin{cases} C_1^A < C_2^A \\ C_1^B > C_2^B \end{cases}$$

Now consider the bundle giving to agent A the expected value of C^A (denoted $(\overline{C}^A)$ in both states, i.e. $(\overline{C}^A, \overline{C}^A)$, where:

$$\overline{C}^A = q_1^A \times C_1^A + (1 - q_1^A) \times C_1^A$$

Clearly agent A's utility is higher at $(\overline{C}^A, \overline{C}^A)$, since, by strict concavity of $U_A(\cdot)$:

$$V^A(P, \overline{C}^A) = u_A(\overline{C}^A) > \mathbb{E}_q u_A(C^A) = V^A(P, C^A)$$

Now consider agent B: by the endowment constraints we know that:

$$\overline{C}_1^A + \overline{C}_1^B = C_1^A + C_1^B$$
$$\overline{C}_2^A + \overline{C}_2^B = C_2^A + C_2^B$$

Therefore:

$$\overline{C}_1^B = C_1^B - q_2^A \underbrace{(C_2^A - C_1^A)}_{>1} < C_1^B$$
$$\overline{C}_2^B = C_1^B + q_1^A \underbrace{(C_2^A - C_1^A)}_{>1} > C_1^B$$

Noticing also that $\overline{C}_1^B > \overline{C}_2^B$, we obtain:

$$C_1^B > \overline{C}_1^B > \overline{C}_2^B > C_2^B \tag{*}$$

We can now prove that $V^B(P,\overline{C}^B) - V^B(P,C^B) > 0$. First recall that, combining (*) with the concavity of $U_B(\cdot)$, we have that:

$$\frac{U_B(\overline{C}_2^B) - U_B(C_2^B)}{\overline{C}_2^B - C_2^B} > \frac{U_B(C_1^B) - U_B(\overline{C}_1^B)}{C_1^B - \overline{C}_1^B} \tag{**}$$

We can express the denominators as follows:

$$\overline{C}_2^B - C_2^B = q_1^A(C_2^A - C_1^A)$$
$$C_1^B - \overline{C}_1^B = (1 - q_1^A)(C_2^A - C_1^A)$$

and simplify equation (**):

$$\frac{U_B(\overline{C}_2^B) - U_B(C_2^B)}{q_1^A\cancel{(C_2^A - C_1^A)}} > \frac{U_B(C_1^B) - U_B(\overline{C}_1^B)}{(1 - q_1^A)\cancel{(C_2^A - C_1^A)}}$$

$$\Leftrightarrow U_B(\overline{C}_2^B) - U_B(C_2^B) > \frac{q_1^A}{(1 - q_1^A)}(U_B(C_1^B) - U_B(\overline{C}_1^B))$$

Finally:

$$\begin{aligned} V^B(P,\overline{C}^B) - V^B(P,C^B) &= -\,p_1^B[(U_B(C_1^B) - U_B(\overline{C}_1^B)] + (1 - p_1^B)[U_B(\overline{C}_2^B) - U_B(C_2^B)] \\ &> -\,p_1^B[(U_B(C_1^B) - U_B(\overline{C}_1^B)] + \left(1 - p_1^B\frac{q_1^A}{1 - q_1^A}\right)[U_B(C_1^B) - U_B(\overline{C}_1^B)] \end{aligned}$$

The RHS can be further simplified:

$$\begin{aligned} &= \left(-p_1^B + (1 - p_1^B)\frac{q_1^A}{1 - q_1^A}\right)[(U_B(C_1^B) - U_B(\overline{C}_1^B)] \\ &= \underbrace{\left(\frac{q_1^A - p_1^B}{1 - q_1^A}\right))}_{>0}\underbrace{[(U_B(C_1^B) - U_B(\overline{C}_1^B)])}_{>0} \end{aligned}$$

The second term is positive since $U_B(\cdot)$ is increasing, while the first is positive because, agent A being more uncertainty averse than agent B:

$$p_1^A < p_1^B \leq q_1^B < q_1^A \Rightarrow q_1^A - p_1^B > 0$$

Hence, $V^B(P, \overline{C}^B) - V^B(P, C^B) > 0$ and (C^A, C^B) is Pareto dominated by $(\overline{C}^A, \overline{C}^B)$. □

4.4 The general case: more than two states

Generalizing the previous analysis to the case of n states of the world raises some unavoidable difficulties, which strongly limit in scope the analysis of efficient trades. Unless some specific subclasses of aggregate endowments are considered, the Pareto locus cannot be clearly characterized. Two related complications prevent the analysis to be general: on one hand the relevant minimizing prior does not depend anymore on the ranking of outcomes only, thereby impeding the analyst to consider "classes" of allocations. On the other hand, kinks in the upper contour set of an agent cease to be uniquely located at the locus of constant consumption allocations. As a consequence, comonotonicity of the consumption plans cannot be guaranteed at an interior optimum and full insurance behaviour becomes unrelated with the degree of uncertainty aversion. In this sense, ε-MEU preferences do not differ much from standard MEU ones: when we cannot identify each agent's minimizing priors the additional structure provided by the model here considered becomes irrelevant. The following examples illustrate.

Example 4. *Impossible identification.* Consider an agent with log utility whose selected PPS is given by:

$$\varphi(P) = \{(\tfrac{1}{4} + \alpha, \tfrac{1}{2} - 2\alpha, \tfrac{1}{4} + \alpha), \alpha \in [0, \tfrac{1}{6}]\}$$

Even though the ranking of the outcomes does not change (the two allocations are comonotone), the two following allocations will be minimized by different priors in $\varphi(P)$:

$$\begin{aligned} \hat{C} = (1,2,3) \quad &\Rightarrow \quad \text{Minimizing prior:} \quad \hat{\pi} = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}) \\ \tilde{C} = 3\hat{C} = (3,6,9) \quad &\Rightarrow \quad \text{Minimizing prior:} \quad \tilde{\pi} = (\tfrac{5}{12}, \tfrac{1}{6}, \tfrac{5}{12}) \end{aligned}$$

Example 5. *Non comonotone P.O. allocations.* The agent of example 4 above (say, i) faces now another agent (j) who is uncertainty neutral and has selected PPS $\varphi^j(P) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in an economy where $\omega = (5, 16, 15)$. Suppose both agents have log utility: $U^i(C_s) = U^j(C_s) = \log(C_s)$. An allocation such that agent i chooses $\tilde{C}^1 = (3,6,9)$ and agent j: $\tilde{C}^j = \omega - \tilde{C} = (2, 10, 6)$ is Pareto optimal.

Example 6. *Non-differentiability at non-constant allocations.* Suppose that there are three states of the world, $S = \{1,2,3\}$, $\omega_1 > \omega_2 > \omega_3$ and agent A's selected PPS takes the form:

$$\varphi^A(P) = \{(\tfrac{1}{4}, \alpha, \tfrac{3}{4} - \alpha), \alpha \in [0, \tfrac{1}{2}]\}$$

The upper contour set of agent A will display a kink at all (non-constant) allocations involving the following consumption pattern:

$$\hat{C}^A = \{(a, b, b), a \in (0, \omega_1), b \in (0, \omega_3)\}$$

In particular, at such allocations all priors in $\varphi^A(P)$ minimize the expected utility of agent A (this is however not necessary in general).

Example 7. *Full insurance.* Consider an economy with three states of the world, $S = \{1, 2, 3\}$, and assume $\omega = (10, 9, 8)$. The PPS is given by:

$$P = \{(\tfrac{1}{4} + \alpha, \tfrac{1}{3}, \tfrac{5}{12} - \alpha), \alpha \in [\tfrac{1}{6}, \tfrac{1}{12}]\}$$

There are three agents, Anne, Benjamin and Carl, with log utility and initial endowment:

$$\omega^A = (2, 2, 2)$$
$$\omega^B = (3, 3, 3)$$
$$\omega^C = (5, 4, 3)$$

Carl is very averse to uncertainty, $\varepsilon^C = 0$, and therefore $\varphi^C = P$; Anne and Bob are also uncertainty averse, but a little less than Carl: $\varepsilon^A > \varepsilon^B > 0$. Since Anne and Benjamin have constant initial endowment, any prior in their selected PPS supports their upper contour sets. We know that, whatever their degree of uncertainty aversion, the Steiner point $s(P) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ always belongs to their set of selected beliefs, meaning that their MRS between any two state can take on the value 1. On the other hand, the prior that minimizes Carl's initial bundle is $\pi^C = (\frac{5}{12}, \frac{1}{3}, \frac{1}{4})$. Hence, the initial endownment is itself Pareto optimal, it involves full insurance for more than one agent, but not for the most uncertainty averse one.

As we just mentioned, some assumption on the aggregate endowment can help to characterize the efficient locus. This is what is done in Chateauneuf and Di Castro (2011) [3], who, analyzing the standard MEU setting, introduce the notion of *unanimously unambiguous* allocation and show that whenever the initial endowment is such, the Pareto locus can be characterized. Moreover, they establish a connection between the P.O. locus in the MMEU economy and (each of) the related economy of SEU maximizers with common prior. Since their result applies here, we shortly summarize it (adjusting the notation to the model here considered) and provide an illustration.

An event is unambiguous for agent i if all priors in i's selected PPS assign to it the same probability. Formally:

Definition 3. *An event $A \in \Sigma$ is unambiguous for agent i if $\pi(A) = \pi'(A)$, $\forall \pi, \pi' \in \varphi^i(P)$. The set of unambiguous events $A \in \Sigma$ is denoted by Σ_u.*

The definition of unambiguous allocation follows straightforwardly: an allocation is unambiguous if it can "decomposed" in unambiguous events, so that its expected value is the same according to all priors in an agent's selected PPS:

Definition 4. *An allocation $C^i \in \mathbb{R}^S$ is unambiguous for agent i if it is Σ_u-measurable in the sense that $\{s \in S : C^i_s \geq t\} \in \Sigma_u$ holds for every $t \in \mathbb{R}$.*
Finally, an allocation $(C^i)_{i=1,\dots,n}$ is unanimously unambiguous if it is unambiguous for all agents $i \in \mathcal{N}$.

Chateauneuf and De Castro prove (Theorem 5.2 in [3]) that if the aggregate endowment $\omega = \sum_{i\in\mathcal{N}} \omega^i$ is unanimously unambiguous, then:

1. the existence of an interior unanimously unambiguous P.O. allocation is guaranteed;
2. any P.O. allocation is unanimously unambiguous;
3. there exists at least one common probability, *conditional* on the unanimously unambiguous sigma-algebra.

Further, they show that the (unanimously unambiguous) Pareto locus can be characterized through its comparison with the P.O. locus of the corresponding vNM economy where agents have the same prior. In particular:

Let $\pi \in \cap_{i\in\mathcal{N}} \varphi^i(P)$. Then $O^u_{MEU} = O_{EU(\pi)} \cap A^u_{MEU}$, where:

- O^u_{MMEU} is the set of unambiguous P.O. allocations in the MMEU economy;
- A^u_{MMEU} is the set of unambiguous allocations in the MMEU economy;
- $O_{EU(\pi)}$ denotes the set of unambiguous P.O. allocations in the economy with expected utility maximizers having common prior π.

Obviously, the notion of unambiguous allocation generalizes and parallels the one of full insurance: recall that full insurance behaviour is likely to be P.O. in an economy with ambiguity aversion because agents are unwilling to trade from a constant bundle to a random one if the two have the same expected value for some prior in their selected PPS. The above theorem characterizes a similar sort of unwillingness to trade: agents never swap a risky bundle with an ambiguous one and the set of admitted Pareto improvements involves only "unambiguously safe" bundles, i.e. bundles with (strictly) greater expected return according to all priors in their selected PPS.

The assumption of unambiguity is imposed on the aggregate endowment because if it is possible to prove that an initial allocation is itself Pareto optimal, a no trade result is equivalent to the characterization of the Pareto locus. Note that allocations involving full insurance for some agent i are unambiguous for agent i but not in general for all other agents. As a consequence, an unanimously unambiguous aggregate endowment typically fail to exist, as the next example highlights.

Example 8. *Inexistence* Suppose that there are two agents, three states of the world, $S = \{1, 2, 3\}$, $\omega_1 > \omega_2 > \omega_3$ and the PPS takes the form:

$$P = \{(\tfrac{1}{3} - \alpha, \tfrac{1}{6} + 2\alpha, \tfrac{1}{2} - \alpha), \alpha \in [0, \tfrac{1}{6}]\}$$

Unless at least one of the two agents (say i) is uncertainty neutral ($\varepsilon^i = 1$), in which case all allocations on j's certainty line are unanimously unambiguous, the aggregate endowment is never unanimously unambiguous in this economy.

Next, let us try to highlight the peculiarity of ε-MMEU preferences:

Example 9. *Allocations that are unanimously unambiguous in a MMEU economy but cannot be such in the ε-MMEU one*
Suppose that there are three states of the world, $S = \{1, 2, 3\}$, $\omega = (5, 4, 3)$. There are two agents, A and B. In the MMEU setting, their set of beliefs is not linked to any objective, and common, information present in the economy and they can therefore "disagree" in their estimations. Suppose that the agents have the following sets of beliefs:

$$\begin{aligned} P^A = & \{(\tfrac{1}{4}, \alpha, \tfrac{3}{4} - \alpha), \alpha \in [0, \tfrac{1}{2}]\} \\ P^B = & \{(\tfrac{3}{4} - \alpha, \alpha, \tfrac{1}{4}), \alpha \in [0, \tfrac{1}{2}]\} \end{aligned}$$

Any aggregate endowment of the type:
$\omega = \{(\omega^A = (a, b, b), \omega^B = (5 - a, 4 - b, 3 - b) | a < 5, b < 4, a > b, (a - b) = 1\}$,
e.g. $\omega = ((3, 2, 2)_A, (2, 2, 1)_B)$ is both unanimously unambiguous and Pareto optimal. However, in the ε-MMEU setting agents are constrained to some sort of agreement, which makes, whenever $\varepsilon^i \neq 1, \forall i = \{A, B\}$, their unambiguous algebras coincide. Hence, in some cases, further restrictions on the aggregate endowment may be required for the existence of unanimously unambiguous P.O allocations. As an illustration, suppose the PPS in the economy is $P = P^A$, $\varepsilon^A = 0$ and $\varepsilon^B \neq 1$. Then no P.O. allocation is unanimously unambiguous since no allocation involving the same consumption levels in state 2 and 3 for both agents can satisfy the endowment constraints (nor can a full insurance allocation).

The previous example highlights the peculiarity of ε-MMEU preferences: provided that $\varepsilon^i \neq 1, \forall i \in \mathcal{N}$, the presence of some objective information induces all agents to have the same unambiguous algebra $\Sigma_u^i = \Sigma_u, \forall i$. Put it differently, unambiguous events are *objectively* so. (As a consequence, also, the intersection of the agents' conditional probabilities is always non-empty: $\cap_{i \in \mathcal{N}} \mathcal{P}_i^{uu} \neq \emptyset$ where: $\mathcal{P}_i^{uu} \equiv \{\pi(\cdot \mid \Sigma_u) : \pi \in \varphi^i(P)\}$. This is due to the fact that the conditional priors are objective too: they depend on the PPS, not on the degrees of uncertainty aversion).

This lead us to study one case, perhaps not very interesting, in which the only consideration ("by eye") of the aggregate endowment tells us whether Pareto-optima are guaranteed to be comonotone: when there is a unique unambiguous partition of the state space[13]. Intuitively, at an unambiguous allocation the agents "hedge" against the uncertainty in the sense of having a consumption which is constant across the states of the world whose union composes an unambiguous event. It follows that, provided that the state space can be uniquely "sliced" in terms of unambiguous events, i.e. if there exists

[13]Note that in principle this is observable, since information is objective. This observability may be useful in applications.

a unique (non-trivial) unambiguous partition of the state space, $\{A^m\}_{m\in\mathcal{M}}$, $A^m \in \Sigma_u$, $\forall m$, the aggregate endowment should also be constant across those states of the world composing some unambiguous event, if it is to be unanimously unambiguous. Formally:

Proposition 4. *Assume that $\varepsilon^i \neq 1, \forall i \in \mathcal{N}$ and that there exists a unique unambiguous partition of the state space $\{A^m\}_{m\in\mathcal{M}}$, $A^m \in \Sigma_u$, $\forall A^m$. Then the following are equivalent:*

- *$\exists$ a P.O. unanimously unambiguous allocation $(C^i)_{i=1,\ldots,n}$;*
- *$\forall s^m_l, s^m_{l'} \in A^m, \omega_l = \omega_{l'}$.*

Proof. (1) $\Rightarrow$ (2) Since $(C^i)_{i=1,\ldots,n}$ is unanimously unambiguous, it is unambiguous for all agents, i.e. $\forall i, C^i$ is Σ_u-measurable. Since, by assumption , there is a unique unambiguous partition of the state space $\{A^m\}_{m\in\mathcal{M}}$, $A^m \in \Sigma_u$, $\forall A^m$, it must be that:

$$\forall i, C^i : \text{unambiguous} \Rightarrow \forall s^m_l, s^m_{l'} \in A^m, C^i_l = C^i_{l'}$$

Hence: $\sum_{i\in\mathcal{N}} C^i_l = C_l = \sum_{i\in\mathcal{N}} C^i_{l'} = C_{l'}$.
(2) $\Rightarrow$ (1) If $\forall s^m_l, s^m_{l'} \in A^m, \omega_l = \omega_{l'}$, then there exists an allocation in which each agent can have the same consumption across the states of the world whose union constitutes an ambiguous event, while satisfying the endowment constraints. This guarantees the existence of an unanimously unambiguous aggregate endowment and, since $\cap_{i\in\mathcal{N}}\mathcal{P}^{uu}_i \neq \emptyset$, by Theorem 5.2 we know that any P.O. allocation is unanimously unambiguous and comonotone. $\square$

The previous proposition does not hold, in general, if there exist more than one unambiguous partition of the state space, as the following example illustrates.

Example 10. Let $S = \{a, b, c, d\}$ and the PPS be given by: $Q = (\frac{\alpha}{2}, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}, \frac{\alpha}{2}), \alpha \in [0, 1]\}$. Clearly, $\Sigma_u = \{\{a, b\}, \{c, d\}, \{a, c\}\{b, d\}\}$, and there are two possible unambiguous partitions of the state space: $A_1 = \{\{a, b\}, \{c, d\}\}$ and $\{\{a, c\}\{b, d\}\}$. The aggregate endowment is $\omega = (4, 5, 6, 7)$, there are two agents, A and B, and their initial endowment is given by: $\omega^A = (1, 1, 3, 3), \omega^A = (3, 4, 3, 4)$. The initial endowment is obviously unanimously unambiguous, comonotone and Pareto optimal.

5 Conclusions

The present analysis confirms two broad facts that have been established in the literature, namely, that uncertainty aversion tends to reduce purely speculative trade and that this reduction is related to some sort of minimal consensus among the agents' probabilistic assessment. Preferences *à la* Gajdos et al. give a realistic explanation for this agreement to be guaranteed, with the additional desiderata of providing a comparative index which is easy to elicit, the objective information being common, and immediately observable.

The reference to some objective set of probability distributions seems to give rise to some sort of efficient ambiguity-sharing behaviour in the two-state case, where the agents' attitudes towards uncertainty affect the optimal arrangements as much as their attitudes towards risk. Interestingly, it does so by precisely "pricing" the disagreement across agents that is reasonable to admit, at least in the sense of being compatible with the information available. Even though the agents' priors differ, the trade agreement is not "spurious", suggesting that perhaps not *all* trades based on some probabilistic disagreement are purely speculative.

Yet, the difficulties related to the analysis of more than two-state choices remain hard to tackle, even if some additional comparability across agents is added, so as to order their beliefs by set inclusion. While the intra- and interpersonal comparability of beliefs is lost when the aggregate endowment is left undisciplined, once unambiguous events are considered the additional structure of ε-MMEU preferences does restore full unanimity in beliefs: the objectivity of the information has a cogent character.

In between these two extrema, a gray area remains unexplored: the additional structure provided by this model of rationality seems to be insufficient to clearly characterize efficient trade, and, at this stage, I am not aware of any condition that could improve the analysis.

References

[1] Billot, A., Chateauneuf, A., Gilboa, I., and Tallon, J.-M. Sharing beliefs: between agreeing and disagreeing. *Econometrica 68*, 3 (2000), 685–694.

[2] Chateauneuf, A., Dana, R.-A., and Tallon, J.-M. Optimal risk-sharing rules and equilibria with choquet-expected-utility. *Journal of Mathematical Economics 34*, 2 (2000), 191–214.

[3] De Castro, L. I., and Chateauneuf, A. Ambiguity aversion and trade. *Economic Theory 48*, 2-3 (2011), 243–273.

[4] Ellsberg, D. Risk, ambiguity, and the savage axioms. *The Quarterly Journal of Economics 75*, 4 (1961), 643–669.

[5] Gajdos, T., Hayashi, T., Tallon, J.-M., and Vergnaud, J.-C. Attitude toward imprecise information. *Journal of Economic Theory 140*, 1 (2008), 27–65.

[6] Gilboa, I., Maccheroni, F., Marinacci, M., and Schmeidler, D. Objective and subjective rationality in a multiple prior model. *Econometrica 78*, 2 (2010), 755–770.

[7] Gilboa, I., and Schmeidler, D. Maxmin expected utility with non-unique prior. *Journal of mathematical economics 18*, 2 (1989), 141–153.

[8] Klibanoff, P., Marinacci, M., and Mukerji, S. A smooth model of decision making under ambiguity. *Econometrica 73*, 6 (2005), 1849–1892.

[9] Rigotti, L., Shannon, C., and Strzalecki, T. Subjective beliefs and ex ante trade. *Econometrica 76*, 5 (2008), 1167–1190.

[10] Strzalecki, T., and Werner, J. Efficient allocations under ambiguity. *Journal of Economic Theory 146*, 3 (2011), 1173–1194.

[11] Wakker, P. P. *Prospect theory: For risk and ambiguity*, vol. 44. Cambridge University Press Cambridge, 2010.

[12] Yaari, M. E. Some remarks on measures of risk aversion and on their uses. *Journal of Economic theory 1*, 3 (1969), 315–329.